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Capacities: From Information Theory to Extremal Set Theory

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Generalizing the concept of zero-error capacity beyond its traditional links to any sort of information transmission we give an asymptotic solution to several hard problems in extremal set theory within a unified, formally information-theoretic framework. The results include the solution of far-reaching generalizations of Rényi's problem on qualitatively independent partitions.

1. INTRODUCTION

Looking through a channel. Given a finite set V, when do we consider two elements of V^n really different? Beyond the obvious first answer, information theory has reproposed this question in the work of Claude E. Shannon. Two sequences are "really different" from our point of view, if we manage to tell the one from the other, and this may depend on the way in which we are looking at them. Shannon proposed to look at them through a noisy channel and thus has created two tremendously difficult problems for the combinatorialist: the code distance problem [14, 20] and the zero-error capacity problem [26]. With both of these problems one is interested in how many pairwise "really different" sequences we can construct for given V and n. In the code distance problem we fix some $\alpha > 0$ and say that $\mathbf{v}$ and $\mathbf{v}' \in V^n$ are r.d. (really different) if they differ in at least αn positions, i.e., if their Hamming distance is at least αn. Shannon proposed to study the exponential behaviour of the largest cardinality of a set of pairwise r.d. sequences as a function of α. The answer is still unknown but the problem has inspired original research even in algebraic geometry [28]. In the binary case $|V| = 2$ the problem

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can be interpreted in the extremal set theory language. One is looking for the largest family of subsets of an n-set every pair of members of which have a symmetric difference of cardinality at least αn. One is not able to decide whether the construction guaranteed by the greedy algorithm [14] is asymptotically optimal. The code distance problem is of great practical interest to the communication engineer and has a vast literature.

The problem of zero-error capacity of the discrete memoryless channel is familiar to the graph theorist in a channel-free formulation. The stochastic description of the channel is translated into purely graph-theoretic terms. Shannon [26] defines an arbitrary graph G on the set V and makes us call two elements of V^n "really different" if among the coordinate pairs $(v_1, v_1'), \ldots, (v_n, v_n')$ of $\mathbf{v} = v_1 \cdots v_n$ and $\mathbf{v}' = v_1' \cdots v_n'$ there is some edge $(a, b) \in E(G)$, i.e., if we have$(v_i, v_i') \in E(G)$ for some $i = 1, \ldots, n$. Let us denote by $N(G, n)$ the largest cardinality of a set $C \subset V^n$ of pairwise r.d. sequences in the present sense. The always existing limit

$$C(G) = \lim_{n\to\infty} \frac{1}{n} \log N(G, n)$$

is the (zero-error) capacity of the (channel associated with the) graph G. Shannon [26] observed that if the chromatic number $\chi(G)$ of the graph G is equal to its clique number (the maximum cardinality of a complete subgraph of G), then $C(G) = \log \chi(G)$. Thus if a graph is perfect [3, 2], then $C'(G') = \log \chi(G')$ for each of its induced subgraphs G'. It is perhaps worthwhile remembering that Claude Berge's motivation in introducing the notion of a perfect graph was indeed information-theoretic. The problem of determining Shannon capacity becomes intriguing for minimally imperfect graphs of which the pentagon, the cycle of length 5, is the smallest and simplest. Its capacity has served as a challenge to many a mathematician for 20 years, until a brilliant and elementary solution was found by Lászlo Lovász [19]. The capacity problem for graphs is still open even for the cycle of length 7.

The practical interpretation of graph capacity is quite simple. The vertex set of the graph represents the input alphabet of a noisy channel that can be used to successively transmit any sequence of letters from the input alphabet. However, the action of the noise effecting the transmission is such that different input letters can result in the same output at the receiving end of the channel. For some letters this may never occur. Whether or not two letters are distinguishable in this sense at the receiving end can be reflected by associating a graph to the channel in which two vertices are joined by an edge if the corresponding letters are distinguishable at the receiving end of the underlying channel. Clearly, two sequences of input letters of the same length can be distinguished if at least in one coordinate the corresponding vertices are joined by an edge in the graph. Hence $N(G, n)$ is the maximum number of pairwise distinguishable sequences of channel inputs; it describes the maximum noiseless transmission capability of the noisy channel the graph has resulted from.

The above account should suffice to convince the reader that information theory has been an important source of inspiration to the combinatorialist. However, it has offered just problems, not solutions, and these problems have turned out to be difficult for anyone to solve. Thus the zero-error capacity problem in particular has functioned as some sort of scarecrow. Nobody seems to be able to solve it and after some initial enthusiasm created by the Lovász result [19] (cf. the papers of Haemers [15], McEliece, Rodemich, and Rumsey [21], and Schrijver [24]), no further progress has been made in this direction in the last 10 years; for a survey cf. Haemers' article in [25]. Understandably, a generalization of the zero-error capacity problem, as proposed by Cohen, Körner, and Simonyi in [7], has seemed meaningless even though it was pointed out that the more general question is in striking similarity with many difficult problems in extremal set theory. Now that we have managed to solve some of these combinatorial problems in [13] (cf. also [17]), we repropose our approach in even more generality and solve an entire class of problems both in combinatorics and in classical Shannon theory. This is done in the present paper by both simplifying and generalizing the approach of our paper [13].

Looking through an unknown channel. In [7], Cohen, Körner, and Simonyi studied a generalization of zero-error capacity.

In the simplest setting, the transmission problems of information theory require the design of codes for a channel of which the stochastic description is available to the code designer. In the case of zero-error capacity this means that the distinguishability properties of the pairs of letters are described by a single graph. To depart from this often unrealistic assumption, information theory is also dealing with the problem of designing codes that fit several channels at the same time. The simplest model of this kind, sometimes called the compound channel (introduced independently by Blackwell, Breiman, Thomasian [4], Dobrushin [10], and Wolfowitz [29]; cf. [9]) asks for the construction of codes that work for any of a finite number of channels having the same input alphabet, provided that the channel actually used is always the same during the transmission. For the details of the stochastic model we refer the reader to [9] or [7], since at present we are only interested in its zero-error case. Once again, the latter can be formulated in a purely combinatorial language as follows:

Let $\mathscr{G}$ be a finite family of graphs with common vertex set V. Let us call the elements of V^n "really different for $\mathscr{G}$" if they are r.d. for every graph $G \in \mathscr{G}$ in the previous sense. Let us denote by $N(\mathscr{G}, n)$ the largest cardinality of a set $C \subset V^n$ of pairwise r.d. sequences in the present sense, i.e., for $\mathscr{G}$. The main problem in [7] was to determine the always existing limit

$$C(\mathscr{G}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathscr{G}, n).$$

The quantity $C(\mathscr{G})$ is called the zero-error capacity of the compound channel associated with $\mathscr{G}$, or, from the combinatorial point of view, the capacity of the family of graphs $\mathscr{G}$. In a way, its determination seems hopeless, since it involves a plain generalization of an unsolved problem, that of the Shannon capacity of a single graph. The main question in [7] was, however, somewhat different. Suppose for a moment that every graph in $\mathscr{G}$ is so simple that the determination of their capacity is a trivial task. Would we then be able to determine the capacity of the family $\mathscr{G}$? The authors of [7] argued that this is a different question the main difficulty of which is "disjoint" from that of finding the Shannon capacity of a single graph. We will show that this is the case, indeed. In [7], a general upper bound on $\mathscr{G}$ was derived. The upper bound (to be stated later in the text) is not always "computable"—but despite this difficulty we show in the present paper that it is always tight. At present, we just want to anticipate this new result by explaining one of its consequences.

"Or" and "and" capacity of a graph. Let us be given a graph G with vertex set V. We have just defined two different capacities associated with this graph. One, the Shannon capacity, can be considered an "or-capacity" for the edge-set of G. In its definition, two elements $\mathbf{v}$ and $\mathbf{v}'$ of V^n are considered "really different" if there is a coordinate i for which $(v_i, v_i') \in E(G)$, i.e., at least *one* of the edges of G occurs among the coordinate pairs (v_i, v_i')—but it can be *either one* of the edges of G; this is what we mean by calling $C(G)$ an or-capacity. What then would an "and-capacity" be like? In its definition, one would require that *every* edge of the graph G be present among the coordinate pairs of the sequences. This definition is clearly a special case of the new notion, Shannon capacity for a family of graphs. In fact, as in [7], let us denote by $\mathscr{F}(G)$ the family of single-edge graphs obtained by considering for every $e \in E(G)$ the graph G_e defined by setting

$$E(G_e) = \{e\}$$
$$V(G_e) = V.$$

Thus $\mathscr{F}(G)$ is the family of the $|E(G)|$ different single-edge graphs we have obtained. One has

THEOREM CKS [7].

$$C(\mathscr{F}(G)) \leq \max_P \min_{(a,b)\in E(G)}[P(a) + P(b)] \times h(P(a)/(P(a) + P(b))),$$

where the maximum is for all probability distributions on the vertex set V of G, and h is the binary entropy function

$$h(t) = -t \log t - (1 - t)\log(1 - t).$$

(Here and in the sequel log's and exp's are binary. For the information-theory background we refer the reader to [9] even though we intend the paper to be self-contained.)

We shall prove among other things that $C(\mathscr{F}(G))$ actually equals the upper bound in Theorem CKS. We can see from the formula that $C(\mathscr{F}(G))$ is determined in terms of the quantities $[P(a) + P(b)]h(P(a)/(P(a) + P(b)))$, which are the values of Shannon capacity of the graphs $G_{(a,b)}$—the graphs with vertex set V and a single edge (a, b)—even though the capacity involved is a refined version of Shannon capacity; it is a merely technical quantity introduced in Csiszár and Körner [8], where it is called capacity within a given type P. This concept plays a central role in this paper. Its formal definition is postponed to the formal part of our text. (Often, it can be expressed in terms of graph entropy, a simple quantity associated with a graph and a probability distribution on its vertex set [16].)

Capacities for directed graphs—Sperner capacity. No question naturally arising in extremal set theory can be formulated in terms of any of the above problems. Yet, as observed in [7, 17], there is a striking similarity between certain questions of the information theorist and some well-known problems in extremal set theory. Therefore, it seemed natural to formulate a framework for extremal set theory that is formally information-theoretic, even though it is lacking any interpretation in terms of transmitting information. In the present paper this approach is presented in its full generality as it was announced in [13].

DEFINITION GKV [12]. Let G be a directed graph with a (finite) set of vertices V and a set of arcs $E(G) \subset V^2$. Note that $(a, b) \in E(G)$ does not exclude $(b, a) \in E(G)$. Let us say that the sequence $\mathbf{v} = v_1 \cdots v_n \in V^n$ precedes the sequence $\mathbf{v}' = v_1' \cdots v_n' \in V^n$ relative to G (this is yet another sense of "really different") if there is some i $(1 \leq i \leq n)$ for which $(v_i, v_i') \in E(G)$.

Let us say that a set $C \subset V^n$ is incomparable for G if for every ordered pair of elements $\mathbf{v} \in C, \mathbf{v}' \in C$ the sequence $\mathbf{v}$ precedes $\mathbf{v}'$. (Thus, in particular, for every pair of sequences $\mathbf{v}, \mathbf{v}'$ with $\mathbf{v} \neq \mathbf{v}'$ in C we see that both $\mathbf{v}$ precedes $\mathbf{v}'$ and $\mathbf{v}'$ precedes $\mathbf{v}$.)

Let $I(G, n)$ be the largest cardinality of a set C in V^n that is incomparable for G. We call

$$\Sigma(G) = \limsup_{n \to \infty} \frac{1}{n} \log I(G, n)$$

the Sperner capacity of G.

Notice that in complete analogy with Shannon capacities, the above lim sup is actually a limit.

Remark. Let G be a directed graph for which $(a, b) \in E(G)$ implies $(b, a) \in E(G)$. (Such a graph G is called symmetric by Berge [2].) Let G' be the corresponding undirected graph. Clearly, the Sperner capacity of G equals the Shannon capacity of G'. Hence, at least formally, the concept of Sperner capacity is more general than that of Shannon capacity.

For some simple graphs we can prove that their Sperner capacity does not depend on the particular orientation of the edges. This raises the following

Problem [12, 13]. Is there any graph G for which $\Sigma(G)$ depends on the particular orientation of the arcs? In particular, if G' is the symmetric graph corresponding to the arbitrary directed graph G (i.e., the minimal symmetric graph containing all the arcs of G), can one ever have

$$\Sigma(G) < \Sigma(G')?$$

Calderbank, Frankl, Graham, Li, and Shepp [6] proved that if G is the cycle on three vertices (with cyclically oriented edges) then $1 = \Sigma(G) < \Sigma(G') = \log 3$.

The reason behind the name Sperner capacity is a classical result of Sperner. Suppose that $V = \{0, 1\}$, and G has the single arc $(0, 1)$. Then the exact value of $I(G, n)$ is the subject of Sperner's theorem [27], who proved that

THEOREM SP. $I(G, n) = \binom{n}{\lfloor n/2 \rfloor}$. *Thus,* $\Sigma(G) = 1$.

Now we are ready to face the subject of this paper, Sperner capacity of graph families. This concept has become easy to guess. It represents a common generalization of all the previous concepts we have mentioned so far. (The idea of this kind of capacities came up in joint work with G. Simonyi [17]; the concept introduced in [17] coincides with ours in special cases but is different in others and will not be treated here.) As we have mentioned, the determination of Sperner capacity in special cases has already allowed us to prove some interesting combinatorial results such as the solution of Rényi's problem concerning the maximum number of pairwise qualitatively independent partitions in [13] (cf. [22] for an account of previous work) and a k-partite Sperner theorem generalizing the 2-partite Sperner theorem of Körner and Simonyi [17]. All this will now follow from a very general new theorem implying many further results in extremal set theory. Before the formal discussion starts, let us briefly anticipate some of these.

Graph-dependent partition systems. The intersection pattern of two k-partitions of an n-set can be described by a directed graph on k vertices in which the vertices of the graph correspond to the classes in the partitions and in which there is an arc from vertex a to vertex b precisely when the class labelled a of the first partition has a non-empty intersection with the class labelled b of the second partition. In [13] we solved two problems that can be described as the determination of the asymptotics of the maximum number of k-partitions of an n-set with the property that every pair of partitions follow the same intersection pattern prescribed by a fixed graph G on k vertices. In fact, if G is the complete directed graph with k^2 arcs between its k vertices, then we get the qualitative independence problem of Rényi mentioned above [23]. The other problem we solved in [13] can be described in this language by means of a star graph. In [13] we could solve such problems only for graphs G exhibiting a strong symmetry. Here we solve them for arbitrary graphs. But our main results will be even more general and will allow us to treat Sperner-type problems in a substantially more general way. In fact, it seems to us that a considerable number of new results in extremal set theory follow from our Theorem 1. We limit ourselves to presenting a few examples here. A systematic study of all the applications will be the subject of further research. The present paper is a continuation of our work in [13], to which we shall sometimes refer.

Recall that log's and exp's are always binary.

2. Capacity of a Family of Directed Graphs

In the rest of this paper we deal with just one notion of capacity. This, however, will be general enough to contain all the previous definitions as a special case.

Definition 1. Let $\mathscr{G}$ be a family of directed graphs, each having the same finite vertex set V. A set $C \subset V^n$ is called incomparable for $\mathscr{G}$ if for every $\mathbf{v}, \mathbf{v}' \in C$ ($\mathbf{v} = v_1, \ldots, v_n, \mathbf{v}' = v'_1, \ldots, v'_n, \mathbf{v} \neq \mathbf{v}'$) and for every $G \in \mathscr{G}$ there is a coordinate i, $1 \le i \le n$, such that $(v_i, v'_i) \in E(G)$. (Thus $C \subset V^n$

is incomparable for $\mathcal{G}$ if it is incomparable for every $G \in \mathcal{G}$ in the sense of Definition GKV.)

Let $I(\mathcal{G}, n)$ denote the largest cardinality of a set $C \subset V^n$ that is incomparable for $\mathcal{G}$. We call

$$\Sigma(\mathcal{G}) = \lim_{n \to \infty} \frac{1}{n} \log I(\mathcal{G}, n)$$

the Sperner capacity of $\mathcal{G}$.

Note that as before the limit always exists since $\log I(\mathcal{G}, n)$ is superadditive.

Our main goal in this paper is to express the Sperner capacity of a family of graphs in terms of some parameters of the individual graphs in the family. (It is not immediately obvious that such a description is possible.) We need some notation.

Given a sequence $\mathbf{x} \in V^n$ we shall denote by $P_{\mathbf{x}}$ the probability distribution on the elements of V defined as

$$P_{\mathbf{x}}(a) = \frac{1}{n} |\{i : x_i = a, i = 1, 2, \ldots, n\}|,$$

where $\mathbf{x} = x_1 \cdots x_n$. $P_{\mathbf{x}}$ is called the type of $\mathbf{x}$. Let $V^n(P, \varepsilon)$ denote the set of those $\mathbf{x} \in V^n$ for which

$$|P_{\mathbf{x}} - P| = \max_{a \in V} |P_{\mathbf{x}}(a) - P(a)| \leq \varepsilon.$$

Let us write $V_P^n = V^n(P, 0)$. For an arbitrary directed graph G, let $I(G, P, \varepsilon, n)$ be the largest cardinality of any set $C \subset V^n(P, \varepsilon)$ which is incomparable for G in the sense of Definition GKV. Write

$$\Sigma(G, P) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log I(G, P, \varepsilon, n).$$

One easily sees that

LEMMA 1. *The Sperner capacity of an arbitrary finite family of directed graphs $\mathcal{G}$ satisfies*

$$\Sigma(\mathcal{G}) \leq \max_P \min_{G \in \mathcal{G}} \Sigma(G, P).$$

Proof. The proof is identical to that of Lemma 1 in [7]. For the sake of completeness, we repeat it.

Clearly, the number of possible types of sequences in V^n is upper bounded by $(n+1)^{|V|}$. Let us denote the family of these types by $\mathcal{P}_n$.

Then, for every $\varepsilon > 0$,

$$V^n = \bigcup_{P \in \mathscr{P}_n} V^n(P, \varepsilon).$$

This means that, for every $\varepsilon > 0$,

$$I(\mathscr{G}, n) \leq |\mathscr{P}_n| \max_{P \in \mathscr{P}_n} \min_{G \in \mathscr{G}} I(G, P, \varepsilon, n).$$

Hence, for every $\varepsilon > 0$,

$$\frac{1}{n} \log I(\mathscr{G}, n) \leq \frac{|V| \log(n+1)}{n} + \max_P \frac{1}{n} \log \min_{G \in \mathscr{G}} I(G, P, \varepsilon, n).$$

Since $|V| < \infty$ and, by super-additivity with respect to n,

$$\limsup_{n \to \infty} \max_P \frac{1}{n} \log \min_{G \in \mathscr{G}} I(G, P, \varepsilon, n) = \max_P \limsup_{n \to \infty} \frac{1}{n} \log \min_{G \in \mathscr{G}} I(G, P, \varepsilon, n)$$

the lemma follows. ▮

The main mathematical content of the present paper is to prove that this upper bound is actually tight. This innocent and very technical-looking statement will have many consequences to which we shall return later.

In order to establish the lower bound counterpart of Lemma 1 we make a technical observation to be used in the proof. Here and in the sequel $|\mathscr{S}|$ denotes the number of non-empty classes of partition $\mathscr{S}$.

LEMMA 2. *Let us have two arbitrary partitions, $\mathscr{S}$ and $\mathscr{T}$ of a finite set X. We can construct new partitions, $\mathscr{S}^*$ and $\mathscr{T}^*$ such that $\mathscr{S}^*$ refines $\mathscr{S}$, $\mathscr{T}^*$ refines $\mathscr{T}$, the new partitions are equivalent in the sense that some bijection of $\mathscr{S}^*$ into $\mathscr{T}^*$ maps every class to a class of equal size, and their number of classes satisfies*

$$|\mathscr{S}^*| = |\mathscr{T}^*| < |\mathscr{S}| + |\mathscr{T}|.$$

Proof. The statement is easily proved by induction on the sum of the number of the classes in the two starting partitions. Actually, we shall prove a formally more general assertion, for this will make induction easier.

Let us have an arbitrary partition $\mathscr{S}$ of the finite set X and an arbitrary partition $\mathscr{T}$ of the finite set Y, where $|X| = |Y|$. We claim that one can construct two new partitions, $\mathscr{S}^*$ of X and $\mathscr{T}^*$ of Y such that $\mathscr{S}^*$ refines $\mathscr{S}$, $\mathscr{T}^*$ refines $\mathscr{T}$, the new partitions are equivalent in the sense that some bijection of $\mathscr{S}^*$ into $\mathscr{T}^*$ maps every class to a class of equal size, X and Y

map the one into the other, and their number of classes satisfies

$$|\mathscr{S}^*| = |\mathscr{T}^*| < |\mathscr{S}| + |\mathscr{T}|.$$

If $|\mathscr{S}| + |\mathscr{T}| = 2$, we have nothing to prove.

Suppose that the statement is true if $|\mathscr{S}| + |\mathscr{T}| < l$ and consider two partitions of the arbitrary sets X and Y with $|\mathscr{S}| + |\mathscr{T}| = l$.

Let us pick a class A of $\mathscr{S}$ and an arbitrary class B of $\mathscr{T}$. Suppose without loss of generality that

$$|A| \leq |B|.$$

If $|A| = |B|$, drop all the elements of A from X and all the elements of B from Y. If $|A| < |B|$, drop all the elements of A from X and drop $|A|$ elements of B from Y in an arbitrary manner. (Thus a particular element can be dropped on one side and kept on the other one.)

Clearly, after all the above indicated elements are dropped, the resulting new ground sets X' and Y' will continue to satisfy $|X'| = |Y'|$ and the new partitions $\mathscr{S}'$ and $\mathscr{T}'$ will yield

$$|\mathscr{S}'| + |\mathscr{T}'| < |\mathscr{S}| + |\mathscr{T}|.$$

By the induction hypothesis we can construct the final partitions $\mathscr{S}^*$ and $\mathscr{T}^*$ having strictly less than $1 + |\mathscr{S}'| + |\mathscr{T}'|$ many classes each. (To obtain them, just add to the common refinement of $\mathscr{S}'$ and $\mathscr{T}'$ the respective classes dropped from X and Y.) ∎

For the sake of self-contained derivation of our central result we prefer to quote in full a computational lemma from [13]. The simple proof is routine application of Jensen's inequality to the function $x \log x$.

LEMMA 3. *Set*

$$f(\mathbf{x}) = f(x_1, \dots, x_A) = \sum_{i=1}^{A} x_i i \log i$$

and consider the inequalities (equalities)

$$\sum_{i=1}^{A} x_i \leq C$$

$$\sum_{i=1}^{A} ix_i = T$$

$$x_i \geq 0.$$

Let B be the set of vectors $\mathbf{x} = (x_1, \dots, x_A)$ which satisfy the three preceding *relations. We have*

$$\min_{\mathbf{x} \in B} f(\mathbf{x}) \geq T \log \frac{T}{C}.$$

Our proof of the lower bound will follow right away from the following:

THEOREM 1. *Let $\mathscr{F}$ and $\mathscr{G}$ be two arbitrary families of directed graphs with the same vertex set V. Then for any probability distribution P on V, we have*

$$\Sigma(\mathscr{F} \cup \mathscr{G}, P) \geq \min\{\Sigma(\mathscr{F}, P), \Sigma(\mathscr{G}, P)\}.$$

Remark. This theorem is a substantial generalization of the Main Corollary in [13]. Its proof, however, is just a readaptation of that of the Main Corollary. As in order to be self-contained we have chosen to give a full proof, it has become impossible to avoid the literal repetition of parts of the corresponding proof from [13]. The only substantial difference is our present reliance on Lemma 2, the new tool that has made this generality possible.

Proof. Let P be a probability distribution on the set V. It is easily seen that the definition of $\Sigma(\mathscr{F}, P)$ and $\Sigma(\mathscr{G}, P)$ actually implies the existence of a sequence P_n of distributions on V such that

$$\max_{a \in V} |P(a) - P_n(a)| \to 0$$

and the corresponding sequences of sets $A_n \subset V_{P_n}^n$ and $B_n \subset V_{P_n}^n$ for which

$$\lim_{n \to \infty} \frac{1}{n} \log |A_n| = \Sigma(\mathscr{F}, P)$$

and

$$\lim_{n \to \infty} \frac{1}{n} |B_n| = \Sigma(\mathscr{G}, P),$$

where the set A_n is incomparable for $\mathscr{F}$ and the set B_n is incomparable for $\mathscr{G}$, for every n.

Let Σ_n be the group of all permutations of $\{1, 2, \ldots, n\}$. For a sequence $\mathbf{a} \in V^n$ and a permutation $\pi \in \Sigma_n$ let us denote by $\pi(\mathbf{a})$ the sequence

$$\pi(\mathbf{a}) = a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(n)}.$$

Likewise, we write

$$\pi(A_n) = \{\pi(\mathbf{a}) : \mathbf{a} \in A_n\}.$$

Obviously,

$$\bigcup_{\pi \in \Sigma_n} \pi(A_n) = V_{P_n}^n$$

and each $\mathbf{a} \in V_{P_n}^n$ belongs to $|A_n|$ sets $\pi(A_n)$. By a well-known theorem of Lovász [18], there exist t permutations, $\pi_1, \pi_2, \ldots, \pi_t \in \Sigma_n$, with

$$t \leq \frac{|V_{P_n}^n|}{|A_n|} \log(2|A_n|) \tag{1}$$

such that for every n, $V_{P_n}^n$ is the disjoint union of sets

$$S_i \subseteq \pi_i(A_n), \qquad 1 \leq i \leq t.$$

(The present special case of Lovász' theorem is explicitly stated in Ahlswede [1] as the Covering Lemma.) Let us denote by t_i the number of those sets S_j which have cardinality i. We have

$$\sum_{i=1}^{|A_n|} it_i = |V_{P_n}^n|. \tag{2}$$

Repeating the above argument for the sequence of sets B_n we can decompose $V_{P_n}^n$ into the disjoint union of the sets

$$T_i \subseteq \rho_i(B_n), \qquad 1 \leq i \leq u,$$

where $\rho_i \in \Sigma_n$, and

$$u \leq \frac{|V_{P_n}^n|}{|B_n|} \log(2|B_n|). \tag{3}$$

At this point we have two partitions of the set $V_{P_n}^n$ to which we apply Lemma 2. By this lemma, there exist two new partitions, $\{C_i\}_{i=1}^z$ and $\{D_i\}_{i=1}^z$ of $V_{P_n}^n$, with the properties

$$C_i \subset \pi_i^*(A_n), \qquad D_i \subset \rho_i^*(B_n)$$

for some permutations $\pi_i^* \in \Sigma_n$, $\rho_i^* \in \Sigma_n$,

$$|C_i| = |D_i| \qquad \text{for every } i, \tag{4}$$

and

$$z \leq 2\max\left\{\frac{|V_{P_n}^n|}{|A_n|}\log(2|A_n|), \frac{|V_{P_n}^n|}{|B_n|}\log(2|B_n|)\right\}, \tag{5}$$

where the last inequality follows from (1) and (3) by Lemma 2.

Next we construct a Markov chain with the set of states $V_{P_n}^n$. The transition matrix $\{W(\mathbf{b}|\mathbf{a})\}_{\mathbf{a},\mathbf{b}\in V_{P_n}^n}$ is defined by

$$W(\mathbf{b}|\mathbf{a}) = \begin{cases} 1/|D_i| & \text{if } \mathbf{b} \in D_i, \mathbf{a} \in C_i \\ 0 & \text{else.} \end{cases} \tag{6}$$

Note that the uniform distribution over $V_{P_n}^n$ is an invariant distribution for this Markov chain. The chain will help us to construct sequences of some larger length that will be incomparable for $\mathscr{F} \cup \mathscr{G}$. The construction is based on the observation that for two sequences $\alpha, \beta \in (V_{P_n}^n)^m \subset V^{nm}$ such that

$$\alpha = \mathbf{a}_1 \cdots \mathbf{a}_m$$
$$\beta = \mathbf{b}_1 \cdots \mathbf{b}_m$$
$$\mathbf{a}_1 = \mathbf{b}_1, \qquad \mathbf{a}_m = \mathbf{b}_m, \qquad \alpha \neq \beta,$$

and $W(\mathbf{a}_i|\mathbf{a}_{i-1}) > 0, W(\mathbf{b}_i|\mathbf{b}_{i-1}) > 0$, $2 \le i \le m$, one easily verifies that α and β are incomparable for both $\mathscr{F}$ and $\mathscr{G}$. In fact, if $\alpha \neq \beta$, then there exists a first index j for which $\mathbf{a}_j \neq \mathbf{b}_j$. Let j_1 $(2 \le j_1)$ be this index. In other words, $\mathbf{a}_j = \mathbf{b}_j$ for $j < j_1$, while $\mathbf{a}_{j_1} \neq \mathbf{b}_{j_1}$. Let i_1 be the index of the class for which

$$\mathbf{a}_{j_1-1} = \mathbf{b}_{j_1-1} \in C_{i_1}.$$

Since $W(\mathbf{a}_{j_1}|\mathbf{a}_{j_1-1}) > 0$, we see that $\mathbf{a}_{j_1} \in D_{i_1}$. Similarly, we have $\mathbf{b}_{j_1} \in D_{i_1}$.

However, we know that $D_{i_1} \subset \rho_{i_1}^*(B_n)$ for some permutation $\rho_{i_1}^* \in \Sigma_n$, whence it follows that α and β are incomparable for $\mathscr{G}$. Likewise, proceeding from the right, we know that since $\alpha \neq \beta$, there must be a last index j for which $\mathbf{a}_j \neq \mathbf{b}_j$. In other words, there is a $j_2 \le m - 1$ for which

$$\mathbf{a}_{j_2} \neq \mathbf{b}_{j_2}, \qquad \mathbf{a}_j = \mathbf{b}_j, \qquad m \ge j > j_2.$$

Let i_2 be the index of the class of the partition $\{D_i\}_{i=1}^z$ to which $\mathbf{a}_{j_2+1} = \mathbf{b}_{j_2+1}$ belongs, i.e., $\mathbf{a}_{j_2+1} = \mathbf{b}_{j_2+1} \in D_{i_2}$. This implies that

$$\mathbf{a}_{j_2} \in C_{i_2}, \qquad \mathbf{b}_{j_2} \in C_{i_2}.$$

Thus, $\mathbf{a}_{j_2}$ and $\mathbf{b}_{j_2}$ are incomparable for $\mathscr{F}$, and hence so are α and β.

It remains to see that in the above manner we can produce sufficiently many sequences. To do this, we need some standard elementary facts about Markov chains; cf. [11].

Consider the stationary Markov chain having state space $V_{P_n}^n$, transition probability matrix (6), and the uniform distribution as invariant distribution. For every integer m and sequences $\mathbf{a} \in V_{P_n}^n, \mathbf{b} \in V_{P_n}^n$ we define the set

$$M_m(\mathbf{a},\mathbf{b}) = \{\mathbf{a}_1 \cdots \mathbf{a}_m : \mathbf{a}_1 = \mathbf{a}, \mathbf{a}_m = \mathbf{b}, W(\mathbf{a}_i|\mathbf{a}_{i-1}) > 0, i = 2,\dots,m\}.$$

We have seen that $M_m(\mathbf{a},\mathbf{b})$ is incomparable for $\mathcal{F} \cup \mathcal{G}$ whenever it contains at least two elements. Since for

$$M_m = \bigcup_{\substack{\mathbf{a} \in V_{P_n}^n \\ \mathbf{b} \in V_{P_n}^n}} M_m(\mathbf{a},\mathbf{b})$$

we have at least one pair $\mathbf{a} \in V_{P_n}^n, \mathbf{b} \in V_{P_n}^n$ such that

$$|M_m(\mathbf{a},\mathbf{b})| \geq \frac{|M_m|}{|V_{P_n}^n|^2}, \tag{7}$$

it remains to lower bound $|M_m|$. To this end, consider the first m random variables, $X_1, X_2, \dots, X_m$, produced by the Markov chain. It is well known in information theory (cf. Lemma 1.4.2 of [9]) that the joint entropy

$$\begin{aligned} H(X_1, X_2, \dots, X_m) = &-\sum \Pr\{X_1 = \mathbf{a}_1, \dots, X_m = \mathbf{a}_m\} \\ &\times \log \Pr\{X_1 = \mathbf{a}_1, \dots, X_m = \mathbf{a}_m\} \end{aligned}$$

of the first m variable of the (first order) Markov chain $\{X_i\}$ satisfies

$$H(X_2|X_1) \leq \frac{1}{m} H(X_1, X_2, \dots, X_m), \tag{8}$$

where

$$\begin{aligned} &H(X_2|X_1) \\ &\quad = -\sum_{\mathbf{x}} \Pr\{X_1 = \mathbf{x}\} \sum_{\mathbf{y}} \Pr\{X_2 = \mathbf{y}|X_1 = \mathbf{x}\} \log \Pr\{X_2 = \mathbf{y}|X_1 = \mathbf{x}\} \\ &\quad = -\sum_{\mathbf{x}} \Pr\{X_1 = \mathbf{x}\} \sum_{\mathbf{y}} W(\mathbf{y}|\mathbf{x}) \log W(\mathbf{y}|\mathbf{x}) \end{aligned} \tag{9}$$

is the conditional entropy of X_2 given X_1, provided that W is the transition probability matrix of the chain. Now observe that M_m is the set of all sequences $\mathbf{v}$ for which $X_1, \ldots, X_m$ assumes value $\mathbf{v}$ with positive probability. Then (cf. Corollary 1.1.1 in [9])

$$H(X_1, X_2, \ldots, X_m) \le \log|M_m|.$$

Comparing this with (8) we conclude that

$$H(X_2|X_1) \le \frac{1}{m}\log|M_m|. \tag{10}$$

Let us denote by z_i the number of classes in the partitions of $V_{P_n}^n$ resulting from Lemma 2 which have i elements. Note that with this notation we can express the entropy of our Markov chain with state space $V_{P_n}^n$ as (cf. (9))

$$H(X_2|X_1) = \frac{1}{|V_{P_n}^n|}\sum_{i=1}^{z}\sum_{\mathbf{a}\in C_i}\log|C_i| = \frac{1}{|V_{P_n}^n|}\sum_{i=1}^{|A_n|} z_i i\log i$$

with

$$\sum_i z_i \le 2\max\left\{\frac{|V_{P_n}^n|}{|A_n|}\log(2|A_n|), \frac{|V_{P_n}^n|}{|B_n|}\log(2|B_n|)\right\}$$

(cf. (5)) and

$$\sum_i iz_i = |V_{P_n}^n|$$

(cf. (2)). If the elementary estimate of Lemma 3 is applied the above relations imply that

$$H(X_2|X_1) \ge \log\min\left\{\frac{|A_n|}{2\log(2|A_n|)}, \frac{|B_n|}{2\log(2|B_n|)}\right\}.$$

The last inequality and (10) give

$$\frac{1}{m}\log|M_m| \ge \log\min\left\{\frac{|A_n|}{2\log(2|A_n|)}, \frac{|B_n|}{2\log(2|B_n|)}\right\}.$$

In conclusion, there exist $\mathbf{a}, \mathbf{b} \in V_{P_n}^n$ with

$$\frac{1}{m}\log|M_m(\mathbf{a},\mathbf{b})| \ge \log\min\left\{\frac{|A_n|}{2\log(2|A_n|)}, \frac{|B_n|}{2\log(2|B_n|)}\right\} - \frac{2}{m}\log|V_{P_n}^n|.$$

Choosing $m = n$ we now observe that

$$\begin{aligned}\Sigma(\mathscr{F} \cup \mathscr{G}, P) &\geq \limsup_{n\to\infty} \frac{1}{n^2}\log|M_n(\mathbf{a},\mathbf{b})| \\ &\geq \limsup_{n\to\infty} \frac{1}{n}\min\{\log|A_n| - \log\log(2|A_n|) - 1, \log|B_n| \\ &\qquad\qquad - \log\log(2|B_n|) - 1\} - \frac{2\log|V|}{n} \\ &\geq \min\left\{\limsup_{n\to\infty}\frac{1}{n}\log|A_n|, \limsup_{n\to\infty}\frac{1}{n}\log|B_n|\right\} \\ &= \min\{\Sigma(\mathscr{F}, P), \Sigma(\mathscr{G}, P)\},\end{aligned}$$

where the last equality holds by the definition of the sequences of sets A_n, B_n. ∎

THEOREM 2. *The Sperner capacity of an arbitrary finite family of directed graphs $\mathscr{G}$ satisfies*

$$\Sigma(\mathscr{G}) = \max_P \min_{G\in\mathscr{G}} \Sigma(G, P),$$

where the maximum is taken over all the probability distributions P on the common vertex set of the graphs in $\mathscr{G}$.

Proof. In view of Lemma 1 it only remains to show that

$$\Sigma(\mathscr{G}) \geq \max_P \min_{G\in\mathscr{G}} \Sigma(G, P).$$

This will follow from the inequality

$$\Sigma(\mathscr{G}, P) \geq \min_{G\in\mathscr{G}} \Sigma(G, P) \qquad \text{for every } P.$$

To verify the last relation, list the graphs in $\mathscr{G}$ in an arbitrary but fixed order and apply Theorem 1 iteratively to the ever increasing pairs of graph families

$$\begin{aligned}&\mathscr{G}_1 = \{G_1\}, \qquad \mathscr{F}_1 = \{G_2\} \\ &\qquad\qquad\vdots \\ &\mathscr{G}_i = \{G_1, \ldots, G_i\}, \qquad \mathscr{F}_i = \{G_{i+1}\}, \qquad i < |\mathscr{G}|. \end{aligned}$$ ∎

3. Graph Dependent Partition Systems

It is hard to appreciate the power of Theorem 2 since it does not provide a computable formula for determining the Sperner capacity of any family of graphs. What it does instead is tell us that all we have to do is treat individual graphs one by one without worrying about their interrelations. Once we are dealing with graphs that are easy to handle on an individual basis, every problem for graph families becomes solvable through Theorem 2. A good illustration of this situation is provided by families of single-edge graphs. First, a technical lemma.

Lemma 4. *Let P be a probability distribution on the set V and let G be any of the directed graphs all the arcs of which have the two endpoints $a, b \in V$. Then*

$$\Sigma(G, P) = [P(a) + P(b)]h\left(\frac{P(a)}{P(a) + P(b)}\right).$$

Proof. It is easily seen (cf. [16]) that $\Sigma(G, P)$ does not exceed the right-hand side. To prove the opposite inequality, cf. Lemma 2 in [13]. ▌

Recall that in the Introduction we associated with any undirected graph G having vertex set V the family of single-edge graphs $\mathscr{F}(G)$ each graph of which has vertex set V and a different (single) edge of G as the only element of its edge set. Let us now define through a literal repetition of that definition (which we will not carry out) the family $\mathscr{F}(G)$ of single-edge (but many arcs)- graphs associated with an arbitrary directed graph G. The individual graphs of the family can now have one or two arcs between the same endpoints according to the number of arcs existing in G between the corresponding endpoints. We have

Corollary 1. *For an arbitrary directed graph G the Sperner capacity of the family of one-edge graphs $\mathscr{F}(G)$ it defines is*

$$\Theta(G) = \Sigma(\mathscr{F}(G)) = \max_{P} \min_{(a,b)\in E(G)} [P(a) + P(b)]h\left(\frac{P(a)}{P(a) + P(b)}\right),$$

where the maximum is taken for all probability distributions on $V(G)$.

Proof. The statement follows directly from Theorem 2 by using Lemma 4 to determine $\Sigma(\{(a, b)\}, P)$ for the single-edge graphs involved. ▌

We will see next that the corollary is all we need to generalize all the results of our previous paper [13] in the sense anticipated there. It also implies that the upper bound of Theorem CKS quoted in the Introduction is tight. More importantly, the corollary allows us to solve the general problem of determining the asymptotics of the largest cardinality of a graph-dependent partition system of an n-set in the sense of the Introduction.

DEFINITION 2. Let G be an arbitrary directed graph with vertex set V and with possible loops at some of its vertices. We shall say that the partitions $\mathscr{P}$ and $\mathscr{Q}$ of a set X ($|X| \geq |V|$) are G-dependent if both $\mathscr{P}$ and $\mathscr{Q}$ have $|V|$ classes, labelled with the vertices of V and the class of $\mathscr{P}$ labelled "a" has a non-void intersection with the class of $\mathscr{Q}$ labelled "b" whenever (a, b) is an arc of G.

Example. Two k-partitions of an n-set are qualitatively independent if they are G-dependent for the complete directed graph G with a loop at each of its vertices.

THEOREM 3. *Let $R(G, n)$ be the maximum number of partitions of an n-set with the property that any ordered pair of them are G-dependent for the directed graph G. We have*

$$R(G, n) = I(\mathscr{F}(G), n) \tag{11}$$

and

$$\limsup_{n \to \infty} \frac{1}{n} \log R(G, n) = \Sigma(\mathscr{F}(G)).$$

Proof. For the fairly obvious correspondence between graph-dependent partitions of an n-set and sequences which are incomparable for a family of single-edge graphs, cf. the proof of Corollary 3 and the Loop Lemma in [13]. ∎

4. MORE SPERNER-TYPE THEOREMS

Some results concerning graph dependent partition systems reinterpret Sperner-type theorems. An example of this is the k-partite Sperner theorem Corollary 5 in [13], which follows directly from the present Theorem 3 when applied to a star graph. However, there are similar problems not reducible to Sperner capacities of families of single-edge graphs. Many of these problems can still be solved by our method. We shall return to a systematic account on these in a subsequent paper. Here we give just one simple example to illustrate the width of scope of our approach.

DEFINITION 3. The subsets (A_i, B_i), $i = 1, \dots, L$, of an n-set X form a Λ-system if

$$A_i \cup B_i \not\subset A_j \cup B_j \qquad \text{for } i \neq j$$
$$A_i \cap B_j = \emptyset \qquad \text{iff } i = j.$$

Let $L(n)$ denote the maximum number of set pairs in a Λ-system of subsets of an n-set.

Not surprisingly, the determination of $L(n)$ represents yet another problem concerning Sperner capacities and we shall be able to calculate the asymptotics of $L(n)$. We need the following technical observation.

LEMMA 5. *Let us consider the graph G defined by*

$$V(G) = \{0,1,2\}$$
$$E(G) = \{(0,1),(0,2)\}.$$

For any probability distribution P on $V(G)$ we have

$$\Sigma(G,P) = h(P(0)).$$

Proof. Obvious. ∎

LEMMA 6. *Consider the graph family $\{F, G\}$ with*

$$V(F) = V(G) = \{0,1,2\}$$
$$E(G) = \{(0,1),(0,2)\}$$
$$E(F) = \{(1,2)\}.$$

We have

$$\Sigma(\{F,G\}) = \max\{q : h(q) = q\} = 0.7\dots.$$

Proof. Immediate from Theorem 2 by Lemmas 4 and 5. ∎

(We note that the corresponding Shannon-capacity problem and upper bound were introduced in Cohen, Körner, and Simonyi [7].)

COROLLARY 2. *The maximum cardinality $L(n)$ of a Λ-system of subsets of an n-set satisfies*

$$\limsup_{n\to\infty} \frac{1}{n} \log L(n) = \max\{q : h(q) = q\}.$$

Proof. Let us consider the n-set $X = \{1, 2, \ldots, n\}$. To any Λ-system $\{A_l, B_l\}_{l=1}^{L}$ we can associate L elements of $\{0, 1, 2\}^n$ by setting

$$\mathbf{x}^{(l)} = x_1^{(l)} \cdots x_n^{(l)}, \qquad \mathbf{x}_i^{(l)} = \begin{cases} 1 & \text{if } i \in A_l \\ 2 & \text{if } i \in B_l \\ 0 & \text{else.} \end{cases}$$

The conditions on the set pairs are equivalent to the fact that the above sequences are incomparable for the graph family $\{F, G\}$ of the lemma. ∎

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